Navigation Functions for Dynamical, Nonholonomically Constrained Mechanical Systems

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Summary. In this review we explore the possibility of adapting first order hybrid feedback controllers for nonholonomically constrained systems to their dynamical counterparts. For specific instances of first order models of such systems, we have developed gradient based hybrid controllers that use Navigation functions to reach point goals while avoiding obstacle sets along the way. Just as gradient controllers for standard quasi-static mechanical systems give rise to generalized "PD-style" controllers for dynamical versions of those standard systems, so we believe it will be possible to construct similar "lifts" in the presence of non-holonomic constraints notwithstanding the necessary absence of point attractors.

1 Introduction

The use of total energy as a Lyapunov function for mechanical systems has a long history [1] stretching back to Lord Kelvin [2]. Unquestionably, Arimoto [3] represents the earliest exponent of this idea within the modern robotics literature, and, in tribute to his long and important influence, we explore in this paper its extension into the realm of nonholonomically constrained mechanical systems.

The notion of total energy presupposes the presence of potential forces arising from the gradient of a scalar valued function over the configuration space. We focus our interest on "artificial cost functions" introduced by a designer to encode some desired behavior as originally proposed by Khatib [4, 5]. However, we take a global view of the task, presuming a designated set of prohibited configurations — the "obstacles" — and a designated set of selected configurations — the "goal," which we restrict in this paper to be an isolated single point. We achieve the global specification through the

introduction of a *Navigation Function* (NF) [6] — an artificial potential function that attains a maximum value of unity on the entire boundary of the obstacle set, and its only local minimum, at zero, exactly on the isolated goal point. Such functions are guaranteed to exist over any configuration space of relevance to physical mechanical systems [7], and constructive examples have been furnished for a variety of task domains [6, 8, 9, 10].

NF-generated controls applied to completely actuated mechanical systems force convergence to the goal from almost every initial condition and guarantee that no motions will intersect the obstacle set along the way. In the dynamical setting, where the role of kinetic energy is important, they achieve a pattern of behavior analogous to that of similarly controlled corresponding quasi-static dynamics. For example, in the one degree of freedom case, the dynamical setting is represented by the familiar spring-mass-damper system

$$m\ddot{q} + c\dot{q} + kq = 0 \tag{1}$$

and the corresponding quasi-static model arises through a neglect of the inertial forces, $m \to 0$ in (1), yielding

$$c\dot{q} + kq = 0 \tag{2}$$

To illustrate the nature of NF-gradient-based controllers in this simple setting, take the configuration space to be $\mathcal{Q} := \{q \in \mathbb{R} : |q| \leq 1\}$ with navigation function $\varphi(q) := \frac{1}{2}kq^2$, implying that $\{0\} = \varphi^{-1}[0]$ is the goal and $\{-1, 1\} = \varphi^{-1}[1]$ the obstacle set. We imagine that both systems, (1), (2), arise from application of the NF-gradient control law, $u := -\nabla\varphi$, to the respective open loop,

$$\begin{aligned} u &= m\ddot{q} + c\dot{q} \\ &\text{or} \\ u &= c\dot{q}. \end{aligned}$$

We observe that φ is a global Lyapunov function for (2) guaranteeing that all initial conditions give rise to motions that avoid the obstacle set while converging asymptotically on the goal set. Analogously, the total energy, $\mu := \frac{1}{2}\dot{q}^2 + \varphi(q)$ is a Lyapunov function for the velocity-limited extension of $\mathcal{Q}$, $\mathcal{X} := \mu^{-1}[0, 1] = \left\{(q, \dot{q}) \in \mathbb{R}^2 : \mu(q, \dot{q}) \leq 1\right\}$. This guarantees that all initial conditions in $\mathcal{X}$ give rise to motions that avoid the obstacle (and, in fact, are repelled from the entire boundary, $\mu^{-1}[1]$) while converging asymptotically on the zero velocity goal set, $\mu^{-1}[0] = \{(0, 0)\}$. A more general global version of Arimoto's [3] adaptation of Lord Kelvin's observations has been presented in greater detail in [11].

In contrast, for incompletely actuated mechanical systems, when the degrees of freedom exceed the number of independently controlled actuators, the applicability of NF-gradient-based controllers to either dynamical or quasi-static mechanical systems remains largely unexplored. One particularly important class of such systems arises in the presence of nonholonomic

constraints — systems with intrinsically unavailable velocities whose absence cannot be expressed in terms of configuration space obstacles [12]. In such settings, there is an inherent degree of underactuation, since no amount of input work can be exerted in the forbidden directions. In an echo of Arimoto's [3] "lift" of first order gradient dynamics (2) to damped second order mechanical dynamics (1), this paper explores the relationship between quasi-static and fully dynamical NF-gradient controllers for a class of nonholonomic systems.

One very important general observation about nonlinear systems that throws a shadow on every aspect of this exploration was made two decades ago by Brockett [13] who pointed out that nonlinear systems may be completely controllable while failing to be smoothly stabilizable. Nonholonomically constrained systems suffer this defect, so that no smooth feedback controller, NF-gradient or otherwise, could ever stabilize a single point goal. However, switching controllers incur no such limitation.

In the next section we will introduce a switching controller for a broad class of systems that alternately runs "down" and then "across" the gradient slope to bring all motions arbitrarily close to the goal without hitting any obstacles. For these classes we can prove this analytically. The next section addresses the same class of systems, now cast in the dynamical setting. We show how to recast the hybrid controller to give the analogous result for these second order systems. Finally, as an illustrative example, we present numerical simulations of the rolling disk defined in a configuration space with a simple sensory model that imposes a particular configuration space topology.

2 Hybrid Controller for Nonholonomic Kinematic Systems

We start by presenting a class of controllers defined in $\mathbb{R}^3$ for nonholonomic kinematic systems. For an in depth exposure please see [14]. Consider the class of smooth and piecewise analytic, three degree of freedom, drift-free control systems

$$\dot{q} = B(q)u, \quad q \in \mathcal{Q} \subset \mathbb{R}^3; \quad u \in \mathbb{R}^2, \tag{3}$$

where $\mathcal{Q}$ is a smooth and piecewise analytic, compact, connected three dimensional manifold with a boundary, $\partial\mathcal{Q}$ (that separates the acceptable from the forbidden configurations of $\mathbb{R}^3$), possessing a distinguished interior *goal point*, $q^* \in \mathcal{Q}$. In this section we will impose very general assumptions on B and construct a hybrid controller that guarantees local convergence to an arbitrarily small neighborhood of the goal state while avoiding any forbidden configurations along the way.[3]

[3] In the next section, we will introduce more specialized assumptions that extend the basin of attraction to include almost every initial configuration in $\mathcal{Q}$.

We find it convenient to write (3) using the *nonholonomic projection matrix* [15], H into the image of B:

$$H(q) = B(q)B(q)^{\dagger} = B(q)\left(B(q)^T B(q)\right)^{-1} B(q)^T \tag{4}$$

$$\dot{q} = H(q)v, \quad q \in \mathcal{Q} \subset \mathbb{R}^3; \quad v \in \mathbb{R}^3 \tag{5}$$

2.1 Two Controllers and Their Associated Closed Loop Dynamics

It is useful to compare the unconstrained system $\dot{q} = v$ with the constrained version (5). Let φ be a navigation function defined in $\mathcal{Q}$. For the input $v = -\nabla\varphi$ the unconstrained system is globally asymptotically stable at the origin. Using φ as a control Lyapunov function yields $\dot{\varphi} = -\|\nabla\varphi\|^2$. Given this result, a naive approach to attempt stabilizing system (5) is to use the same input $v = -\nabla\varphi$. Define the vector field $f_1 : \mathcal{Q} \to T\mathcal{Q}$ such that $f_1(q) := -H(q)\nabla\varphi(q)$ and the system

$$\dot{q} = f_1(q) = -H(q)\nabla\varphi(q) \tag{6}$$

Since H has a 1-dimensional kernel it follows that (6) has a 1 dimensional center manifold

$$\mathcal{W}^c := \{q \in \mathcal{Q} : H(q)\nabla\varphi(q) = 0\},$$

as corroborated by explicitly computing the Jacobian of f_1 at q^*:

$$Df_1\big|_{q^*} = -(\underbrace{\nabla\varphi\big|_{q^*}}_{=0} \otimes I)DH^S - HD^2\varphi = -\left. HD^2\varphi\right|_{q^*} \tag{7}$$

Using φ as a control Lyapunov function, La Salle's invariance theorem states that system (6) has its limit set in $\mathcal{W}^c$:

$$\begin{aligned}\dot{\varphi} &= -\nabla\varphi^T H \nabla\varphi \\ &= -\|H\nabla\varphi\|^2 \begin{cases} = 0 \text{ if } q \in \mathcal{W}^c \\ < 0 \text{ if } q \notin \mathcal{W}^c \end{cases}\end{aligned} \tag{8}$$

Figure 1 illustrates the topology associated with (6): the projection H imposes a co-dimension 1 foliation complementary to the center manifold. The *stable manifold,* $\mathcal{W}^s$, is the leaf containing the goal, q^*. The input

$$u_1 := B(q)^{\dagger}\nabla\varphi(q) \tag{9}$$

alone cannot stabilize system (6) at the origin, since no smooth time invariant feedback controller has a closed loop system with an asymptotically stable equilibrium point [12]. Nevertheless, for any initial condition outside $\mathcal{W}^c$ an infinitesimal motion in the direction of f_1 reduces the energy φ. If there can be found a second controller that "escapes" $\mathcal{W}^c$ without increasing φ then it

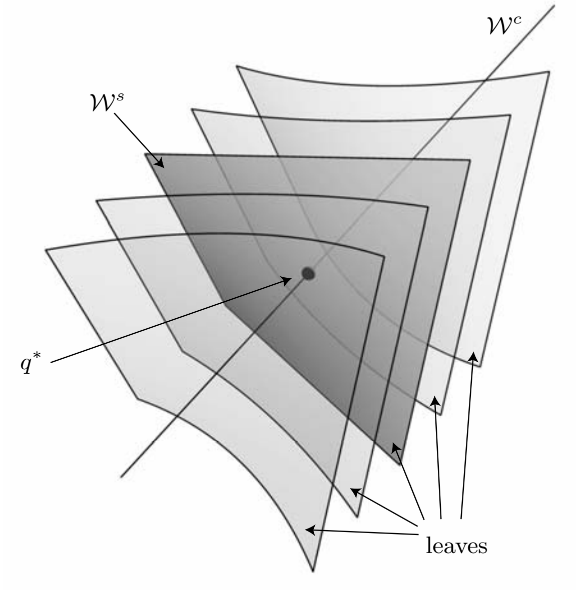

Fig. 1. Conceptual illustration of the flow associated with (6). Each leaf is an invariant manifold with all trajectories collapsing into $\mathcal{W}^c$.

is reasonable to imagine that iterating the successive application of these two controllers might well lead eventually to the goal. We now pursue this idea by introducing the following controller,

$$u_2 := B(q)^\dagger \left[A(q) \times \nabla\varphi(q)\right], \tag{10}$$

leading to the closed loop vector field

$$\begin{aligned} \dot{q} &= f_2(q) \\ f_2(q) &:= A(q) \times \nabla\varphi(q) \end{aligned} \tag{11}$$

where $A(q) := \times B(q)$ is the cross product of the columns of $B(q)$. [4] Note that the nonholonomic constraint expressed in (3) can be represented by the implicit equation $A^T(q)\dot{q} = 0$. Since the derivative of φ in the direction of f_2 is

$$L_{f_2}\varphi = \nabla\varphi(q)^T \cdot (A(q) \times \nabla\varphi(q)) = 0, \tag{12}$$

it follows that f_2 is φ-invariant — i.e. the energy, φ, is constant along its motion. Moreover $0 = A^T(A \times \nabla\varphi) = A^T f_2$, verifying that f_2 indeed satisfies the constraint (3).

[4] We will assume in A3 that B has rank two at each point.

2.2 Assumptions, a Strategy, and Preliminary Analysis

Given the previous two vector fields — one which is energy decreasing; the other energy conserving — we now sketch a strategy that brings initial conditions of system (3) to within an arbitrarily small neighborhood ϵ of the goal, by way of motivating the subsequent definitions and claims that arise in the formal proofs to follow. Let $f_1^t(q)$ and $f_2^t(q)$ denote the flows of f_1 and f_2 respectively.

(1). If $q_0 \in \mathcal{W}^c$ then follow a direction in $\mathrm{im}(H)$ for a finite amount of time t_0 such that $f_1^{t_0}(q_0) \notin \mathcal{W}^c$ and $\varphi \circ f_1^{t_0}(q_0) < 1$ for all $t \in (0, t_0)$.

(2). If $q_0 \notin \mathcal{W}^c$ and $\varphi(q_0) > \epsilon$
  2.1) Use a scaled version of f_2 for time τ_2 to escape a δ-neighborhood of $\mathcal{W}^c$, keeping the energy φ constant.
  2.2) Use controller f_1, for time τ_1, to decrease the energy φ, stopping at a γ-neighborhood of $\mathcal{W}^c$ such that $f_1^{\tau_1}(q) \notin \mathcal{W}^c$ and $\gamma < \delta$.

We now introduce a number of assumptions, definitions and their consequences that will allow us to formalize each of the previous steps:

A1 $\mathcal{Q}$ is a smooth compact connected manifold with boundary.
A2 φ is a navigation function in $\mathcal{Q}$.
A3 H has rank two, uniformly throughout $\mathcal{Q}$.

Assumption A1 gives the proper setting for the existence of a navigation function in the configuration space.Assumption A3 assures the foliation sketched in figure 1.

Define the *local surround* of the goal to be the closed "hollow sphere", $\mathcal{Q}_s := \varphi^{-1}[\Phi_s]$, with $\Phi_s := [\epsilon, \varphi_s]$ whose missing inner "core" is the arbitrarily small open neighborhood, $\mathcal{Q}_\epsilon := \varphi^{-1}[\Phi_\epsilon]$; $\Phi_\epsilon := [0, \epsilon)$, and whose outer "shell", $\mathcal{Q}_1 := \varphi^{-1}[\Phi_1]$, with $\Phi_1 := (\varphi_s, 1]$, includes the remainder of the free configuration space. φ_s is defined to be the largest level such that all the smaller levels, $\varphi_0 \in (0, \varphi_s)$ are homeomorphic to the sphere, S^2, and are all free of critical points, $\|\nabla\varphi\|^{-1}[0] \cap \varphi^{-1}[(0, \varphi_s)] = \emptyset$.

The restriction to φ-invariant topological spheres precludes limit sets of f_2 more complex than simple equilibria in the local surround. However, has in the examples section of [14], one can provide more specialized conditions resulting in the guarantee that the algorithm brings almost every initial condition in the "outer" levels, $\mathcal{Q}_1$ into the local surround, $\mathcal{Q}_s$ and, thence, into the goal set $\mathcal{Q}_\epsilon$.

Lemma 1 ([14]). *Given the previous assumptions*

$$f_1^{-1}[0] \cap \mathcal{Q}_s \equiv f_2^{-1}[0] \cap \mathcal{Q}_s \equiv \mathcal{W}^c \cap \mathcal{Q}_s. \tag{13}$$

To formally express the "δ-neighborhood" described in the stabilization strategy we start by defining the function $\xi : \mathcal{Q} - \{q^*\} \to [0,1]$:

$$\xi(q) := \frac{\|H(q)\nabla\varphi(q)\|^2}{\|\nabla\varphi(q)\|^2} \tag{14}$$

The quantity $\|H(q)\nabla\varphi(q)\|^2$ evaluates to zero only in $\mathcal{W}^c$. Therefore in a small neighborhood of $\mathcal{W}^c$ the level sets of $\|H(q)\nabla\varphi(q)\|^2$ define a "tube" around $\mathcal{W}^c$. The denominator of (14) normalizes ξ such that $0 \leq \xi \leq 1$. Moreover it produces a "pinching" of the tube at the goal q^*.

Lemma 2 ([14]). *For all $\varphi_0 \in \Phi_s$, $\varphi^{-1}[\varphi_0]$ intersects the unit level set of ξ, i.e., $\xi^{-1}[1] \cap \varphi^{-1}[\varphi_0] \neq \emptyset$.*

Corollary 1 ([14]). *For all $\varphi_0 \in \Phi_s$ the level set $\varphi^{-1}[\varphi_0]$ intersects every level set of ξ, i.e., $\xi^{-1}[\alpha] \cap \varphi^{-1}[\varphi_0] \neq \emptyset$ for all $\alpha \in [0,1]$.*

Lemma 3 ([14]). *A sufficient condition for the Jacobian of $f_2(q)$ evaluated at $\mathcal{W}^c - \|\nabla\varphi\|^{-1}[0]$ to have at least one eigenvalue with non-zero real part is that the control Lie algebra on B spans $\mathbb{R}^3$.*

Lemma 4 ([14]). *The Jacobian of $f_2(q)$ evaluated at $\mathcal{W}^c \cap \mathcal{Q}_s$ has two non-zero real part eigenvalues with the same sign.*

Now consider the implicit equation,

$$\xi(q) = \xi^* \Leftrightarrow \|H(q)\nabla\varphi(q)\|^2 = \xi^*\|\nabla\varphi(q)\|^2 \tag{15}$$

At the goal any ξ^* satisfies (15). Although ξ is not defined at q^* all of its level sets intersect at q^*.Finally, define the parameterized cone $\mathcal{C}_\gamma$ around $\mathcal{W}^c$, and its complement $\mathcal{C}_\gamma^c := \mathcal{Q} - \mathcal{C}_\gamma - \{q^*\}$, by:

$$\mathcal{C}_\gamma = \{q \in \mathcal{Q} - \{q^*\} : \xi(q) \leq \gamma\} \tag{16}$$

We follow by imposing conditions on H and A such that the vector field f_2 can afford the needed "escape" from $\mathcal{W}^c$.

Lemma 5 ([14]). *Suppose system (3) satisfies assumptions A1-A3 and, hence, the previous lemmas. Then, there exists a function $\sigma : \mathcal{Q} \to \mathbb{R}$ that renders the system*

$$\dot{q} = \sigma(q)A(q) \times \nabla\varphi(q) = \bar{f}_2(q) \tag{17}$$

unstable at $\mathcal{W}^c \cap \mathcal{Q}_s$.

Corollary 2 ([14]). *Under the conditions of the previous lemma, there can be found a $\tau \in (0,\infty)$ such that for all $q_0 \in \xi^{-1}[\delta/2]$ we have $\xi \circ \bar{f}_2^\tau(q_0) \geq \delta$.*

Figure 2 illustrates the steps used in the previous proof. Trajectories starting inside $\mathcal{N} - \mathcal{C}_\gamma^c$ will traverse $\partial\mathcal{C}_\gamma$ and $\partial\mathcal{C}_\delta$ in finite time.

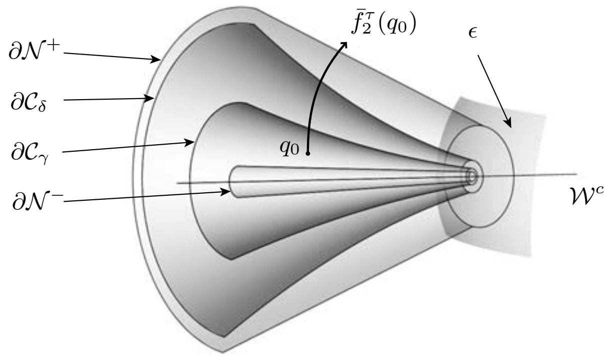

Fig. 2. Illustration of the construction used in the proof of corollary 2.

2.3 A Hybrid Controller and Proof of its Local Convergence

Given the previous result define the time variables τ_1, τ_2 and the scalars $\gamma < \delta$ such that:

$$\tau_1(q,\gamma) := \begin{cases} \min\{t > 0 \mid \xi(f_1^t(q)) = \gamma\} & \text{if } q \in \mathcal{C}_\gamma^c \\ 0 & \text{otherwise} \end{cases}$$
$$\tau_2(q,\delta) := \begin{cases} \min\left\{t > 0 \mid \xi(\bar{f}_2^t(q)) = \delta\right\} & \text{if } q \in \mathcal{C}_\delta - \mathcal{W}^c \\ 0 & \text{otherwise} \end{cases}$$

I.e., τ_1 is the time to reach the γ neighborhood of $\mathcal{W}^c$ using vector field f_1 and τ_2 is the time to escape from a γ neighborhood to a δ neighborhood of $\mathcal{W}^c$ using vector field $\bar{f}_2$.

This results in the following maps:

$$f_1^{\tau_1} : \overline{\mathcal{C}_\gamma^c} \to \partial\mathcal{C}_\gamma \tag{18}$$
$$\bar{f}_2^{\tau_2} : \mathcal{Q}_s - \mathcal{W}^c \to \overline{\mathcal{C}_\delta^c} \subset \overline{\mathcal{C}_\gamma^c}, \tag{19}$$

where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$. With $\delta = 2\gamma$ define the map $P : \mathcal{Q}_s - \mathcal{W}^c \to \partial\mathcal{C}_\gamma$

$$P(q) = f_1^{\tau_1(\cdot,\gamma)} \circ \bar{f}_2^{\tau_2(q,2\gamma)}(q) \tag{20}$$

and consider the recursive equation:

$$q_{k+1} = P(q_k). \tag{21}$$

The set $\partial\mathcal{C}_\gamma$ can be interpreted as a Poincaré section for the discrete system (21). We are now ready to present the following result:

Theorem 1 ([14]). *There exists an iteration number, $N : \mathcal{Q}_s \to \mathbb{N}$ such that the iterated hybrid dynamics, P^N brings $\mathcal{Q}_s$ to $\mathcal{Q}_\epsilon$.*

Proof. Define

$$N := \min \{n \in \mathbb{N} | 0 \leq N \leq N_\epsilon | \varphi \circ P^n(q_0) \leq \epsilon\},$$

and $\Delta\varphi(q) := \varphi \circ P(q) - \varphi(q)$. Since $\mathcal{Q}_s$ is a compact set it follows that $|\Delta\varphi|$ achieves its minimum value, Δ_ϵ, on that set, hence at most $N_\epsilon := ceiling(\varphi_s - \epsilon)/\Delta_\epsilon$ iterations are required before reaching $\mathcal{Q}_\epsilon$.

Note that all initial conditions in the pre-image of the "local surround", $\mathcal{R} := \bigcup_{t>0} f_1^{-t}(\mathcal{Q}_s - \mathcal{W}^c)$ are easily included in the basin of the goal, $\mathcal{Q}_s$, by an initial application of the controller u_1. While it is difficult to make any general formal statements about the size of $\mathcal{R}$, we show in the next section that for all the examples we have tried, the "missing" initial conditions, $\mathcal{Q} - \mathcal{R} = \mathcal{Z}$, comprise a set of empty interior (in all but one case $\mathcal{Z}$ is actually empty) because all of $\mathcal{W}^c$, excepting at most a set of measure zero, is included in $\mathcal{Q}_s$. In configuration spaces with more complicated topology, there is no reason to believe that this pleasant situation would prevail. To summarize, the following algorithm is guaranteed to bring all initial configurations in $\mathcal{R}$ to the goal set, $\mathcal{Q}_s$:

(1). $\forall q_0 \in \mathcal{Q}_s - \mathcal{W}^c$, follow successive applications of (21), i.e. use the inputs to equation (3):

$$u_1(q) := B^\dagger(q)\nabla\varphi(q) \tag{22}$$

$$u_2(q) := \sigma(q)B^\dagger(q)J(A(q))\nabla\varphi(q) \tag{23}$$

(2). $\forall q_0 \in \mathcal{W}^c$ use the input

$$u_3 := \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \tag{24}$$

for a small amount of time t_3 such that $\varphi \circ f_3^{t_3}(q_0) < 1$, with $f_3(q) := B(q)u_3$.

(3). $\forall q_0 \in \mathcal{R} - \mathcal{Q}_s$, use the input u_1 for time t until $f_1^t(q_0) \in \mathcal{Q}_s$.

2.4 Other Considerations

- **Limit cycles in the level sets of φ.** In many practical applications switching between controllers f_1 and f_2 using a small δ-neighborhood is far too conservative. It may be possible to escape $\mathcal{W}^c$ by more than just the small collar $\xi^{-1}[\delta]$. If we could recognize the passage into $\mathcal{W}^s$ and switch off controller u_2 (i.e. turn $\mathcal{W}^s$ into an attractor of a suitable modified form of f_2) then a final application of controller u_1 is guaranteed to achieve the goal state, q^*. The hope of reworking the form of u_2 so that the resulting closed loop vector field, f_2, has its forward limit set solely in $\mathcal{W}^s$ thus raises the question of when there exists limit cycles in the level sets of φ

for the flow of f_2. More importantly, we seek a condition that guarantees that every trajectory of f_2 starting in a small neighborhood of $\mathcal{W}^c$ can intersect $\mathcal{W}^s$ either by forward or inverse time integration of system (11). Note that f_2 generates a planar flow, making the Bendixon's criteria a natural candidate for such condition. Several authors [16, 17, 18, 19] have developed extensions to Bendixson's criteria for higher dimensional spaces, obtaining in general conditions that preclude invariant sub-manifolds on some set. For systems with first integrals, such as some classes of systems that result from nonholonomic constraints, the conditions simplify to a divergence style test. Feckan's theorem (see [16]) states that in open subsets where $\mathrm{div} f_2 \neq 0$ there can exist no invariant submanifolds of any level precluding cyclic orbits. The divergence measure can be used this way to detect limit cycles. Note however, that the previous result does not preclude quasi-periodic orbits.

- **Computational heuristic substitutes for** σ. The σ function introduced in Lemma 5 modifies the flow of f_2 rendering the center manifold unstable. Having that property is sufficient for stabilization, but more can be accomplished. By careful craft of σ one can minimize the number of switches between controllers f_1 and $\bar{f}_2$ necessary to reach the desired neighborhood of the goal. If the stable manifold $\mathcal{W}^s$ is contained in the zero set of σ and $\mathcal{W}^s$ is made attractive by $\bar{f}_2$ for any point in $\mathcal{Q}_s$ then one gets $f_1^\infty \circ \bar{f}_2^\infty(\mathcal{Q}_s) = q^*$, i.e., only 2 steps are necessary to reach the goal. Different methods for approximating σ are presented in [14]. Specifically the function σ is replaced by the divergence of f_2 in a neighborhood of $\mathcal{W}^c$; by maximizing ξ, since that implies escaping $\mathcal{W}^c$ in some measure; or replacing σ by and implicit polynomial stable manifold approximation (please see [20, 14] for invariant manifold computations).

3 Hybrid Controller for Nonholonomic Dynamic Systems

In this section we look into the "lift" of the algorithm proposed in the previous section to nonholonomically constrained dynamical systems. The resulting corollaries arise naturally from the ideas introduced in [21]. Let (25) and (26) be the system of equations for unconstrained systems [22] and nonholonomically constrained systems [23] with $q, u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m, m < n$:

$$M(q)\ddot{q} + c(q, \dot{q}) = u \tag{25}$$

$$\begin{aligned} M(q)\ddot{q} + c(q, \dot{q}) &= A(q)^T \lambda + B(q)v \\ A(q)\dot{q} &= 0 \end{aligned} \tag{26}$$

where M is the mass matrix, c the Coriolis term, A and B represent the actuation constraints defined in section 2 and λ is a vector of Lagrange multipliers.

We start by recalling some notation and lemmas required for the subsequent proofs. Using the "stack-kronecker notation" [24, 25, 26] consider the following linear map:

$$\acute{M}_q : x \mapsto [x \otimes I]^T D_q M^S \tag{27}$$

and the skew-symmetric value operator:

$$J_q(x) := \acute{M}_q(x) - \acute{M}_q^T(x) \tag{28}$$

Lemma 6 ([21]). *For any curve, $q : \mathbb{R} \to \mathcal{Q}$, and any vector, $x \in T_{q(t_0)}\mathcal{Q}$,*

$$\dot{M}_q|_{t_0} x = \acute{M}_{q(t_0)}(x)\dot{q}|_{t_0} \tag{29}$$

Lemma 7 ([21]). *Given a Lagrangian with kinetic energy, κ, with no potential forces present, and with an external torque or force actuating at every degree of freedom as specified by the vector, τ, the equations of motion may be written in the form:*

$$M(q)\ddot{q} + c(q, \dot{q}) = \tau \tag{30}$$

where

$$c(q, x) = C(q, x)x \tag{31}$$

and

$$C(q, x) := \frac{1}{2}\dot{M}(q_x) - \frac{1}{2}J_q(x) \tag{32}$$

Notice that the representation of the Coriolis and centripetal forces in terms of the bilinear operator valued map C only coincide at $\dot{q}$ with the quadratic expression $c(q, \dot{q})$. In general they are not the same.

Corollary 3 ([21]). *For any motion $q : \mathbb{R} \to \mathcal{Q}$, and any tangent vector, $x \in T_{\mathcal{Q}}q(t)$,*

$$x^T \left[\frac{1}{2}\dot{M}(q) - C(q, \dot{q})\right] x \equiv 0 \tag{33}$$

Proof. From the previous lemma,

$$x^T \left[\frac{1}{2}\dot{M}(q) - C(q, \dot{q})\right] x = -\frac{1}{2}x^T J_q(\dot{q})x = 0$$

3.1 Embedding the Limit Behavior of Gradient Dynamics

Controller $f_1(q) = -H(q) \cdot \nabla\varphi(q)$, introduced in Section 2.1, aims to reach a fixed point in the center manifold $\mathcal{W}^c$. In order to lift the controller into a 2nd order system, theorem 2, concerning limit sets of gradient dynamics, is complemented with corollary 4. Let the state variables p_1, p_2 represent $q, \dot{q}$ respectively and let $\mathcal{P} = \mathcal{TQ}$ be the tangent bundle of $\mathcal{Q}$ for system (25).

Theorem 2 (Koditschek[21]). *Let φ be a Morse function on $\mathcal{Q}$ which is exterior directed on the boundary $\partial\mathcal{Q}$, surpasses de value $\mu > 0$ on the boundary, and has a local minima at the points $\mathcal{G} := \{q_i\}_{i=1}^n \subset \mathcal{Q}$. Let $K_2 > 0$ denote some positive definite symmetric matrix. Consider the set of "bounded total energy" states*

$$\mathcal{P}^\mu := \left\{ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathcal{P} : \varphi(p_1) + \frac{1}{2} p_2^T M p_2 \leq \mu \right\} \tag{34}$$

Under the feedback algorithm

$$u := -K_2 p_2 - D\varphi^T(p_1) \tag{35}$$

$\mathcal{P}^\mu$ is a positive invariant set of the closed loop dynamical system within which all initial conditions excluding a set of measure zero take $\mathcal{G}$ as their positive limit set.

Let H be the nonholonomic projection matrix. Define $Q(q) := I - H(q)$ to be the nonholonomic converse projection matrix. Notice that $ker(A) = ker(Q)$ and therefore $Q(q)\dot{q} = 0$. Let $\mathcal{W}_0^c := \{q \in \mathcal{W}^c \wedge \dot{q} = 0\}$. As shown in Section 2.1, $\mathcal{W}^c$ is the center manifold of the system $\dot{q} = -H(q) \cdot \nabla\varphi(q)$. Rewriting equation (26) with a new input $v := B(q)^\dagger u$ we get:

$$\begin{aligned} M(q)\ddot{q} + c(q, \dot{q}) &= A(q)^T \lambda + H(q) u \\ A(q)\dot{q} &= 0 \end{aligned} \tag{36}$$

Corollary 4. *Let $K_2 = \bar{K}_2 H(q)$ with $\bar{K}_2 > 0$ denoting a positive definite symmetric matrix. Under the conditions of theorem 2 all the initial conditions of the system (36), excluding a set of measure zero, take $\mathcal{W}_0^c$ as their positive limit set.*

Proof. Let $V = \varphi(q) + \frac{1}{2}\dot{q}^T M(q)\dot{q}$ be a Lyapunov function for (5). Then

$$\begin{aligned} \dot{V} &= D\varphi\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}\dot{q} - \dot{q}^T \left(HK_2\dot{q} + HD\varphi^T + HC\dot{q} \right) + \underbrace{\dot{q}^T A^T \lambda}_{=0} \\ &= \underbrace{D\varphi Q\dot{q}}_{=0} + \underbrace{\dot{q}^T J_q \dot{q}}_{=0} - \dot{q}^T HK_2\dot{q} \\ &= -\dot{q}^T H^T \bar{K}_2 H\dot{q} \end{aligned}$$

$H^T \bar{K}_2 H$ is a semi-definite positive matrix. $\dot{V}$ is null when either $\dot{q} = 0$ or $H\dot{q} = 0$. Since $A(q)\dot{q} = 0 \Rightarrow H\dot{q} \neq 0$ then the largest invariant set is the interception of the previous sets with $\mathcal{W}_0^c$ resulting in $\mathcal{W}_0^c$. La Salle's theorem guarantees that (5) with input (35) takes $\mathcal{W}_0^c$ as the forward limit.

3.2 Embedding of More General Dynamics

We now seek to lift the controller $f_2(q)$ defined in Section 2.1 to a 2nd order system. We do so by adding once again a level regulator term, so that the reference dynamics attracts to a particular level set. First recall the embedding of general reference dynamics: let f be a reference vector field with Lyapunov function μ, and let $F(p) := p_2 - f(p_1)$. Consider the control algorithm,

$$u = -K_2 F - D\mu^T + MDfp_2 + Cf \tag{37}$$

which applied to the mechanical system (25) yields a closed loop form, $\dot{p} = h(p)$,

$$h(p) := \begin{bmatrix} p_2 \\ Dfp_2 - M^{-1}\left[K_2F + CF + D\mu^T\right] \end{bmatrix}$$

Theorem 3 (Koditschek[21]). *If μ is a strict Lyapunov function for f on $\mathcal{Q}$, then*

$$V := \mu + \frac{1}{2}F^T MF$$

is a strict Lyapunov function for h on $\mathcal{P}$.

In system (5) the set of images of H for each point on $\mathcal{Q}$ is the tangent bundle of $\mathcal{Q}$. Therefore, since H is a projection operator then $\forall x \in \mathcal{P}$ we have $H(p_x).x = x$. Define $\bar{H} = M^{-1}HM$ and $\bar{Q} = M^{-1}QM$. For system (36) with input (37) the closed loop is written in the following way:

$$h(p) := \begin{bmatrix} p_2 \\ \bar{H}Dfp_2 - M^{-1}\left[HK_2F + HCF + HD\mu^T + QCp_2 - A^T\lambda\right] \end{bmatrix}$$

Corollary 5. *Suppose $Hf = f$, i.e. the reference vector field respects the nonholonomic constraints. If μ is a strict Lyapunov function for f on $\mathcal{Q}$ for system (36), then*

$$V := \mu + \frac{1}{2}F^T MF$$

is a strict Lyapunov function for h on $\mathcal{P}$.

Proof. First note that $Q.f = 0;\ Q.p_2 = 0;\ A.f = 0$.

$$\begin{aligned}\dot{V}=&D\mu\ p_2 + \frac{1}{2}F^T\dot{M}F + F^TM\bar{Q}Dfp_2 + \\ &-F^T\left(HK_2F + HCF + HD\mu^T + QCp_2\right) + F^TA^T\lambda \\ =&D\mu Hf - F^THK_2F - \underbrace{f^TQ(MDf\dot{q} - Cf)}_{=0} + \underbrace{f^TA^T\lambda}_{=0} \\ =&D\mu f - F^TK_2F\end{aligned}$$

According to the hypothesis, the first term is negative except on the largest invariant set of $F = 0$. The second term is always negative except in $F^{-1}[0]$. The interception of the two results in the limit set of $\dot{q} = f(q)$.

The previous result show that as long as the reference dynamics respects the nonholonomic constraints we can apply Theorem 3 directly. Notice that Corollary 5 also applies to controller $-H(q).\nabla\varphi(q)$. In general such restrictive dynamics are not necessary for that controller, so using Corollary 4 gives a better tool since we are only interested on the limit set.

4 Simulations

In this section we present simulation examples for the unicycle or vertical rolling disk depicted in Figure 3. The unicycle is commonly defined in the $\mathsf{SE}(2)$ configuration space with constraint equation $\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$. Since we are interested in simulations in a dynamic setting we will follow instead Bloch's vertical rolling disk [27], defined in the configuration space $\mathcal{Q} = \mathbb{R}^2 \times S^1 \times S^1 = \mathsf{SE}(2) \times S^1$ with coordinates $q = (x, x, \theta, \phi)$. The equations of motion for the vertical rolling disk are:

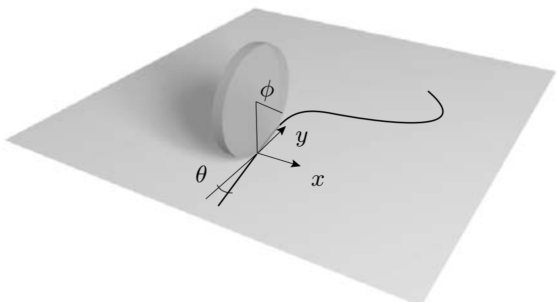

Fig. 3. The vertical rolling disk.

$$
\begin{aligned}
(mR^2 + I)\ddot{\phi} &= u_1 \\
J\ddot{\theta} &= u_2,
\end{aligned}
\tag{38}
$$

with the constraint equations:

$$
\begin{aligned}
\dot{x} &= R\cos(x)\dot{\phi} \\
\dot{y} &= R\sin(x)\dot{\phi}.
\end{aligned}
\tag{39}
$$

Differentiating (39) in time and replacing $\ddot{\phi}$ from (38), one obtains a complete set of equations of motion that verifies the nonholonomic constraints for initial conditions that also verify (39):

$$
\begin{aligned}
\ddot{x} &= -R\sin(\theta)\dot{\theta}\dot{\phi} + \frac{R\cos(\theta)}{mR^2 + I}u_1 \\
\ddot{y} &= R\cos(\theta)\dot{\theta}\dot{\phi} + \frac{R\sin(\theta)}{mR^2 + I}u_1
\end{aligned}
\tag{40}
$$

This system is now written in the form $M(q)\ddot{q} + c(q, \dot{q}) = B(q)u$ for which corollaries 4, 5 apply directly. The $A(q)$ and $B(q)$ matrices are:

$$
A := \begin{bmatrix} -\frac{I + mR^2}{R} & -\sin(\theta) \\ 0 & \cos(\theta) \\ 0 & 0 \\ \cos(\theta) & 0 \end{bmatrix}; \qquad B := \begin{bmatrix} \frac{R\cos(\theta)}{mR^2 + I} & 0 \\ \frac{R\sin(\theta)}{mR^2 + I} & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}
\tag{41}
$$

Although the algorithms presented in section 2 are defined only in $\mathbb{R}^3$, by close inspection of A and B one realizes that, for this particular example, by choosing a $\mathbb{R}^4$ navigation function defined only by the first three parameters of q we will obtain the "same" controller has in $\mathbb{R}^3$. Let φ be a navigation function such that $\nabla\varphi = [\varphi_x, \varphi_y, \varphi_\theta, 0]^T$. Next compute the $\mathbb{R}^4$ cross product of A and $\nabla\varphi$:

$$
\times(A, \nabla\varphi) = \sum_{i,j,k,l=0}^{4} \epsilon_{ijkl} \nabla\varphi_j A_{k1} A_{l2} \hat{e}_i \tag{42}
$$

$$
= \frac{-1}{\cos(\theta)} \begin{bmatrix} \varphi_\theta \cos(\theta) \\ \varphi_\theta \sin(\theta) \\ -\varphi_x \cos(\theta) - \varphi_y \sin(\theta) \\ -\frac{I + mR^2}{R}\theta \end{bmatrix}, \tag{43}
$$

where ϵ_{ijkl} denotes the permutation tensor, $\hat{e}_i$ are the canonical basis vectors and $A_{i,j}$ is the ith row, jth column of A. We now compare with the function

f_2, defined in equation (11) for the $\mathbb{R}^3$ unicycle with $A_3 = [-\sin(\theta), \cos(\theta), 0]^T$ and $\nabla\varphi_3 := [\varphi_x, \varphi_y, \varphi_\theta]^T$:

$$f_2 = A_3 \times \nabla\varphi_3 = \tag{44}$$

$$= \begin{bmatrix} \varphi_\theta \cos(\theta) \\ \varphi_\theta \sin(\theta) \\ -\varphi_x \cos(\theta) - \varphi_y \sin(\theta) \end{bmatrix} \tag{45}$$

The two previous computations produce, in effect, the same behavior for the variables x, y and θ. For each fixed coordinate ϕ in $\mathcal{Q} \subset \mathbb{R}^4$ one obtains a copy of the topology of $\mathsf{SE}(2)$. Therefore, from here on, although the configuration space is defined in $\mathbb{R}^4$, we will only be interested in x, y and θ.

4.1 Navigation Function

Kantor and Rizzi [28] solved the problem of positioning a robot in relation to a single engineered beacon by using the notion of Sequential Composition of Controllers [29]. The final approach to the goal is implemented using Ikeda's Variable Constraint Control. We recast the problem with a NF-encoding according to the approach described in the previous sections, to recover in simulation behavior comparable to that obtained in [28].

Let h be a change of coordinates from $\mathsf{SE}(2) \times S^1$ to double polar coordinates times S^1 that we denote by $\mathcal{P}$ with coordinates $p = [\eta, \mu, d, \phi]^T$:

$$\begin{bmatrix} \eta \\ \mu \\ d \\ \phi \end{bmatrix} = h(x, y, \theta, \phi) = \begin{bmatrix} \arctan(y/x) \\ \theta - \arctan(y/x) \\ \sqrt{x^2 + y^2} \\ \phi \end{bmatrix} \tag{46}$$

Obstacles are introduced on the field of view so that the robot maintains a range of distances to the beacon and keeps facing it:

$$\mu_m < \mu < \mu_M; \qquad d_m < d < d_M \tag{47}$$

Consider the following potential function:

$$\bar{\varphi}(p) := \frac{\left(2 - \cos(\eta - \eta^*) - \cos(\mu - \mu^*) + (d - d^*)^2\right)^k}{(1 - \cos(\mu - \mu_m))(1 - \cos(\mu - \mu_M))} \cdot \\ \cdot \frac{1}{(d - d_m)(d_M - d)} \quad , \tag{48}$$

and its "squashed" navigation function version $\check{\varphi} : \mathcal{P} \to [0, 1]$:

$$\check{\varphi}(p) := \frac{\bar{\varphi}(p)^l}{\epsilon + \bar{\varphi}(p)^l} \tag{49}$$

The navigation function written in the $\mathcal{Q}$ coordinates is $\varphi(q) = \check{\varphi} \circ h(q)$ and its derivative:

$$\nabla\varphi(q) = Dh^T(q) \cdot \nabla\check{\varphi} \circ h(q) \tag{50}$$

We choose to present the Kantor-Rizzi example as the canonical illustration of our ideas due to the interesting topology of the configuration space. Since $\mathcal{Q}$ is not simply connected the level sets of φ change from topological spheres close to the goal q^* to topological tori close to the boundary of $\mathcal{Q}$. Initial conditions stating in the tori will generate quasi-periodic orbits when f_2 is used. In the dynamical setting this provides a good example of the applicability of corollary 5, resulting in the generation of reference dynamics that attract to a particular level set.

4.2 Kinematic Rolling Disk

We first simulate the previously described system in a kinematic setting by solving the system:

$$\dot{q} = B(q)u, \tag{51}$$

and using the control functions f_1, f_2 defined in section 2:

$$u_1(q) := f_1(q) = -H\nabla\varphi \tag{52}$$

$$u_2(q) := \sigma(q)f_2(q) = \times(A, \nabla\varphi)\sigma \tag{53}$$

Figure 4 illustrates the resulting simulation where the initial condition is $q_0 = [1, 1, -\frac{3\pi}{4}, 0]^T$, the desired goal is $q_0 = [0, -2, \frac{\pi}{2}, 0]^T$, the body parameters are $I = J = m = R = 1$, the obstacles are $\mu_m = -\frac{\pi}{4}; \mu_M = \frac{\pi}{4}$; $d_m = 1$; $d_M = 3$ and $\sigma(q) = x$. The manifold $\{q \in \mathcal{Q} : x = 0\}$ is a good local approximation for the stable manifold $\mathcal{W}^s$ of the system $\dot{q} = f_1(q)$. One can observe that from the initial time to t_s the controller f_2 keeps the energy constant while moving exactly in the level set $\varphi^{-1}[\varphi^*]$, with $\varphi^* = 0.98$. At time t_s we switch to controller f_1 and the resulting final position is very close to the goal. Looking at ϕ in the "positions" graphic one observes that the robot does a back and forward motion, necessary to the parallel park maneuver. This comes as a natural consequence of moving in the surface of the torus shown in the "trajectories" plot.

4.3 Dynamic Rolling Disk

For the dynamic setting we solve the system defined by equations (38) and (40):

$$M(q)\ddot{q} + c(q, \dot{q}) = H(q)u, \tag{54}$$

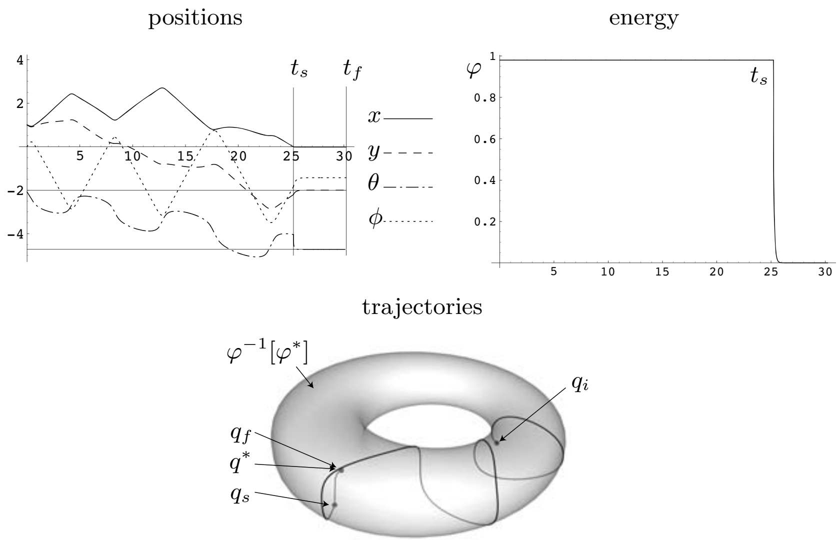

Fig. 4. Kinematic simulation of the vertical rolling disk.

with control functions (35) and (37):

$$u_1(q, \dot{q}) := -K_2\dot{q} - \nabla\varphi \tag{55}$$

$$u_2(q, \dot{q}) := -K_2F - D\mu^T + MDf_2\dot{q} + Cf_2, \tag{56}$$

where $F(q, \dot{q}) := \dot{q} - f_2(q)$ and $\mu := \alpha(\varphi - \varphi^*)^2$.

The first simulation, depicted in Figure 5, uses a high gain $\alpha = 5000$ in the function μ to track the level set as close as possible while the controller f_2 is in use. That results in a good tracking but very jerky steering motion, visible in the first part of the "velocities" and "trajectories" plots. The damping matrix K_2 is set to the identity matrix, resulting in low damping, as observed in the intervals $[t_s, t_f]$ of the "positions" and "velocities" plots.

For the second simulation in the dynamic setting, depicted in Figure 6, the parameter $\alpha = 250$ provokes a less accurate tracking of the desired level set φ^*, when using f_2, as one can observe in the "energy" and "trajectories" plots. However, the resulting motion is smoother then the previous simulation. For the controller f_1, the damping matrix $K_2 = 10I$ slows down the approach to the desired goal, elimination any oscillations as seen in the "energy" plot.

The damping matrix K_2, and the Lyapunov function μ are the design parameters for the control of equation (54).

5 Conclusions

This exploratory discussion paper addresses the reuse of navigation functions developed for fully actuated bodies in the setting of nonholonomic constrained

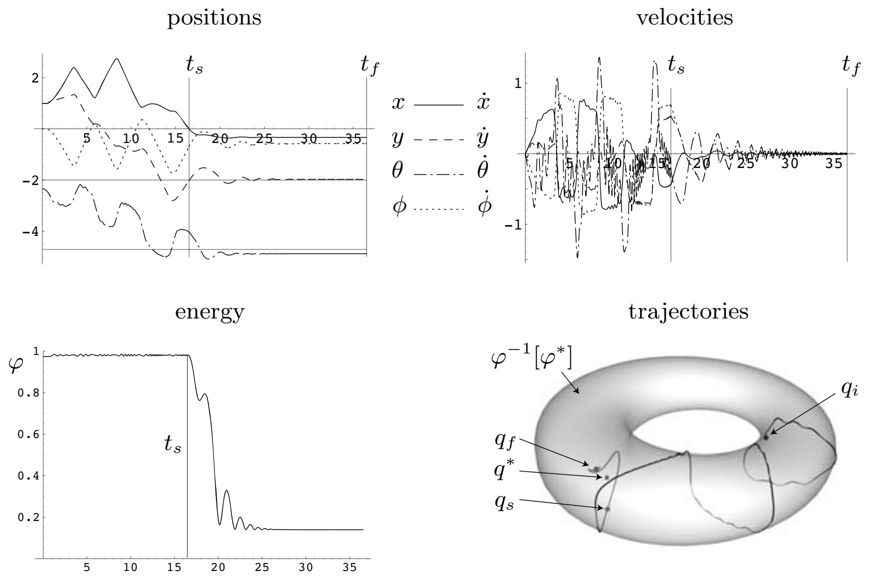

Fig. 5. Dynamic simulation of the vertical rolling disk.

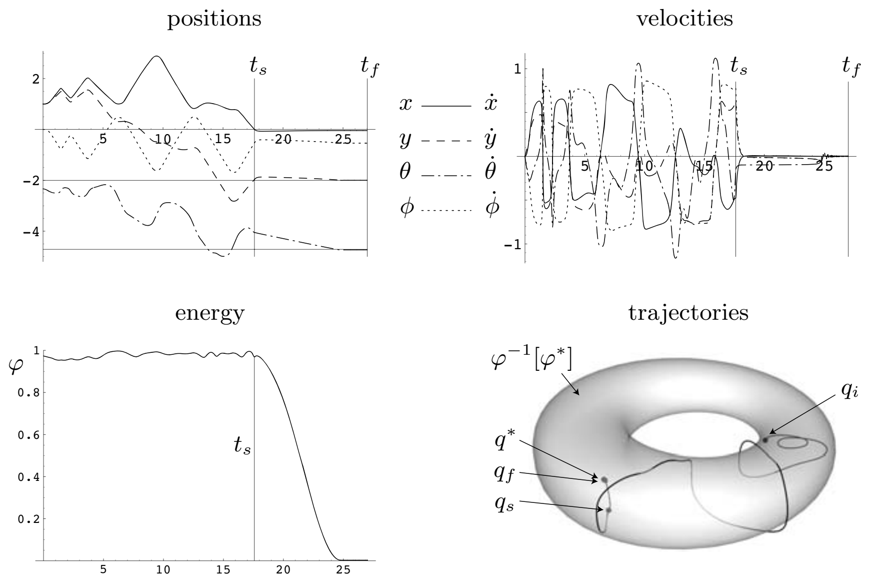

Fig. 6. Dynamic simulation of the vertical rolling disk.

systems for both kinematic and dynamic versions of the model. We suggest how the vector fields developed for kinematic systems can be lifted to the dynamic setting with the introduction of damping and proportional gain type constants. The simulations suggest this lifting can be readily realized in real applications, by proper choice of the damping and gain.

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